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MERCATOR ARTICLE

MERCATOR PROJECTION ON THE SPHERE, A DEDUCTION WITHOUT MATHEMATICAL GAP

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Abstract

Map projection is the mathematical process of converting the Earth's surface, considered as a sphere or an ellipsoid, into a map. This conversion is performed by projecting the Earth's points onto a surface, which can be a plane, a cone, or a cylinder. Its basic objective is to develop a mathematical basis for creating maps, essential in areas such as cartography, geodesy, and navigation. It would be ideal if all maps were isometric, but for large areas, the curvature of the Earth makes it impossible, causing distortions. For the reasons above, the mathematics behind map projection is complex, but it is important to understand it. Among the most varied types, the Mercator projection, created by Gerard Mercator in 1569, is a conformal cylindrical projection, widely used in navigation, as it represents the rhumb lines on the map as straight lines, but, despite preserving angles, it generates other distortions. The objective of this article is to present a complete mathematical derivation of the Mercator projection on the sphere, avoiding simplifications and omissions as much as possible. As an application, the deduced equations will be used to implement a visualization of the continents in Python.

Keywords: Mathematical Cartography, Mapping, Cylindrical Conformal Projection.

Resumo / Resumen

PROJEÇÃO DE MERCATOR NA ESFERA, UMA DEDUÇÃO SEM LACUNAS MATEMÁTICAS

Projeção cartográfica é o processo matemático de conversão da superfície da Terra, considerada como uma esfera ou um elipsoide, em um mapa. Essa conversão é realizada projetando pontos da Terra sobre uma superfície, que pode ser um plano, um cone, ou um cilindro. Assim, seu objetivo básico é criar uma base matemática para a elaboração de mapas, essenciais em áreas como cartografia, geodesia e navegação. Seria ideal que todos os mapas fossem isométricos, porém, para grandes áreas, a curvatura da Terra gera distorções. Pelas razões acima, a matemática das projeções cartográficas é complexa, mas é importante compreendê-la. Entre os mais variados tipos existentes, a projeção de Mercator, criada por Gerard Mercator em 1569, é uma projeção cilíndrica conforme, muito usada em navegação, pois representa no mapa as linhas de rumo como linhas retas, mas, apesar de conservar ângulos, gera outras distorções. O objetivo deste artigo é apresentar uma derivação matemática a mais completa possível da projeção de Mercator na esfera, com o propósito de evitar ao máximo simplificações e omissões, e, como aplicação, utilizar as equações deduzidas para implementar em Python uma visualização dos continentes.

Palavras-chave: Cartografia Matemática, Mapeamento, Projeção Cilíndrica Conforme.

PROYECCIÓN DE MERCATOR SOBRE LA ESFERA: UNA DEDUCCIÓN SIN LAGUNAS MATEMÁTICAS

La proyección cartográfica es el proceso matemático de convertir la superficie de la Tierra, considerada como una esfera o un elipsoide, en un mapa. Esta conversión se realiza proyectando puntos de la Tierra sobre una superficie, que puede ser un plano, un cono o un cilindro. Así, su objetivo es crear una base matemática para la creación de mapas, imprescindible para la cartografía, geodesia y navegación. Sería ideal que todos los mapas fueran isométricos, sin embargo, para áreas grandes, la curvatura de la Tierra genera distorsiones. Por las razones expuestas, las matemáticas de las proyecciones cartográficas son complejas, pero es importante comprenderlas. Entre los varios tipos que existen, la proyección Mercator, creada por Gerard Mercator en 1569, es una proyección cilíndrica conforme, muy utilizada en navegación, ya que representa las líneas de rumbo en el mapa como líneas rectas, pero, a pesar de conservar los ángulos, genera otras distorsiones. El objetivo de este artículo es presentar una derivación matemática la más completa posible de la proyección de Mercator sobre la esfera, con el fin de evitar al máximo simplificaciones y omisiones, y, como aplicación, utilizar las ecuaciones deducidas para implementar una visualización de los continentes en Python.

Palabras-clave: Cartografía Matemática, Mapeo, Proyección Cilíndrica Conforme.

INTRODUCTION

Consider the Earth as a sphere or an ellipsoid. Its three-dimensional surface must be converted into a flat map. This process of mathematical conversion is known as a map projection (KENNEDY and KOPP, 2001). In more technical terms, the shape of the Earth is typically described as a solid of revolution, either an ellipsoid or a sphere, serving as a reference surface to which all physical points are associated. The projection of these points occurs onto a projection surface, which can be of three types: the plane, the cone and the cylinder, with the latter two being capable of unfolding into a plane or flat map (RICHARDUS and ADLER, 1972). Thus, the aim of the study of map projection aims to create a mathematical basis for making maps, and consequently work on solving theoretical and practical problems in cartography, geodesy, geography, astronomy, navigation, and other related sciences (FRANČULA, 2004).

Ideally, a world map would be isometric, meaning it would depict terrain at a reduced scale where measured distances correspond directly to real-world distances through the scale. It works well for small areas but becomes impossible for larger regions due to the Earth's curvature, leading to the preference for plane maps over globes, which are often impractical (VERMEER and RASILA, 2020). It means that a perfect correspondence between the reference surface and the projection surface is impossible due to scale changes and distortions: a 1:1 map of the Earth is unfeasible, and flattening its curved surface inevitably causes some deformation (MALING, 1992).

Nonetheless, the mathematical basis of map projections is complex, which can be intimidating. While those familiar with calculus and differential equations can find detailed books on the subject, most users of geospatial data do not need to know the equations, as mapping software handles the computations. However, having some understanding of the mathematics behind projections is still valuable (KESSLER and BATTERSBY, 2019). In this sense, a lack of understanding of map projections can have serious consequences, such as hindering our grasp of international relations and making us vulnerable to manipulation by politicians, interest groups, and advertisers who may use maps in misleading ways (KIMERLING, MUEHRCKE, and MUEHRCKE, 2005).

Theoretically, there can be an infinite number of map projections (LAPAINE, 2019). One of them, however, stands at the crossroads of a fascinating mix of stories involving navigation, advancements in cartography, military accuracy, media manipulation, and political propaganda: The Mercator projection (MONMONIER, 2004).

In 1569, Gerard Mercator introduced his ‘Ad Usum Navigatorum’ map projection, which is still used for this purpose today (TOBLER, 2017). It is the only conformal cylindrical projection, not distorting angles between intersecting curves (VERMEER and RASILA, 2020). Furthermore, its normal aspect is particularly significant in navigation, as it represents rhumb lines as straight lines (LAPAINE and DIVJAK, 2017). The normal aspect means that the projection wraps a cylinder around the Earth’s reference surface, touching the equator. Hence, meridians are projected as equidistant vertical lines, while parallels are horizontal and mathematically spaced (SNYDER, 1987). In the Mercator projection, the surface of the cylinder is a map, in the sense of the surface unrolled onto the plane (SMETANOVÁ et al., 2016).

In general, the choice of the reference surface for the Earth depends on the purpose of the map (KESSLER and BATTERSBY, 2019), emphasizing that, although the sphere is mathematically simpler to work with compared to the ellipsoid, it also cannot be flattened into a plane without distortions (DIMITRIJEVIĆ, MILOSAVLJEVIĆ and RANČIĆ, 2023). In this sense, for various applications, as in navigation or in thematic mapping with small scales, it is possible to use the sphere as a reference surface for the Mercator projection.

Thus, the aim of this paper is to provide a mathematical derivation as rigorous as possible of the functions that define the Mercator projection on the sphere, minimizing gaps, omissions and simplifications. Furthermore, this approach also has a pedagogical value, since the classical literature on the subject usually presents the equations in their final form or, when derived, often omits intermediate steps, potentially leading to difficulties for those without advanced mathematical training. As an application, this paper includes a visualization of the continental contours, generated through an original implementation in Python, using the derived equations.

CLARIFYING THE PROBLEM

As shown in Figure 1, when using the sphere as a model for the Earth, lines of latitude (parallels) and lines of longitude (meridians) form a reference framework. This set of meridians and parallels on the sphere is called the graticule (OSBORNE, 2013).

Within this framework: a) the north pole (NP) and the south pole (SP) serve as the convergence points for all meridians; b) meridians and parallels intersect perpendicularly; c) all meridians are of equal length and meet at the poles; d) parallels are concentric circles evenly distributed along meridians, and their length decreases as they approach the poles, causing the spacing between meridians along them to range from zero at the poles to a maximum at the equator; e) the equator is the only parallel where the spacing between meridians matches the spacing between all parallels. The process of transferring this system onto a plane surface is the core challenge of map projection (BOWYER and GERMAN, 1959).

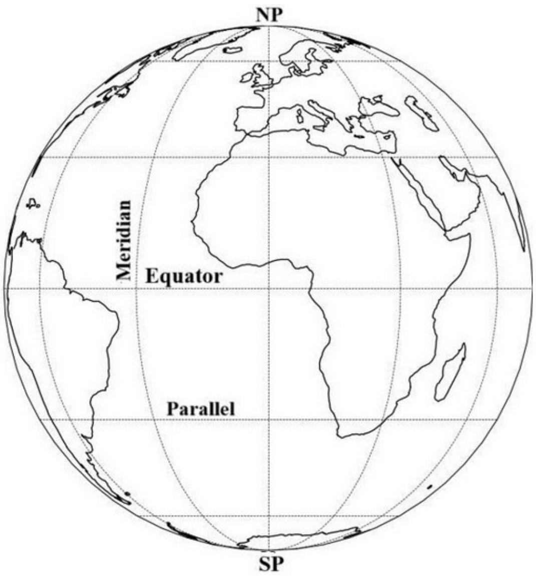

Figure 1 - Lines (or curves) on the surface of the sphere. Source: Authors.

A general mathematical concept of map projection process is given by equations (1) and (2), in which the U and V are to be interpreted as independent coordinates on a projection surface, and φ and λ are to be interpreted respectively as latitude and longitude on a reference surface (TOBLER, 1962).

$$U = F_1(\varphi, \lambda) \quad (1)$$

$$V = F_2(\varphi, \lambda) \quad (2)$$

Functions F1 and F2, known as transformation functions, enable the determination of specific equations for a given map projection using the equations of general concept (BUGAYEVSKIY and SNYDER, 1995).

The general concept leads to a mathematical classification of map projections, known as parametric. The parametric curves on the sphere are the meridians of longitude and parallels of latitude (RICHARDUS and ADLER, 1972). That is,

"The parametric classification is based on the fact that the equations for the location of lines of latitude and of longitude on the map in some cases depend on only one of these quantities. For example, in the cylindrical projections, the lines of longitude depend on longitude alone, and the lines of latitude depend only on latitude. Here it is immediately apparent that the parametric classification tacitly assumes the normal case for each projection. It is convenient to require that the origin of the (U,V) system coincides with the respective origin of the parametric curves of the surface of the sphere. Hence, when the (φ,λ) parameterization is used, the (U,V) coordinates are to be interpreted as rectangular coordinates (X,Y) in the plane" (TOBLER, 1962, p. 168).

Thus, in mathematical terms, for a normal cylindrical projection, equations (1) and (2) become equations (3) and (4) (TOBLER, 1962; BUGAYEVSKIY and SNYDER, 1995):

$$X = F_1(\lambda) \tag{3}$$

$$Y = F_2(\varphi) \tag{4}$$

Clearly, when representing the surface of the Earth onto a plane, the projection must be unique and reversible. In other words, each point on the reference surface should correspond to a single, distinct point on the projection surface, and vice versa (RICHARDUS and ADLER, 1972).

Figure 2 shows that, in Mercator projection, the equator is mapped with coordinates on X axis and the central meridian with coordinates on Y axis (LAPAINE, 2019). Moreover, when the cylinder of the Mercator projection is unfolded, the meridians appear as straight, parallel lines, while the parallels also form straight lines, intersecting them at right angles (RICHARDUS and ADLER, 1972). It should be noted that on Mercator projection, the projected graticule is aligned to the underlying Cartesian system so the constant-x grid lines correspond to meridians running north-south (OSBORNE, 2013).

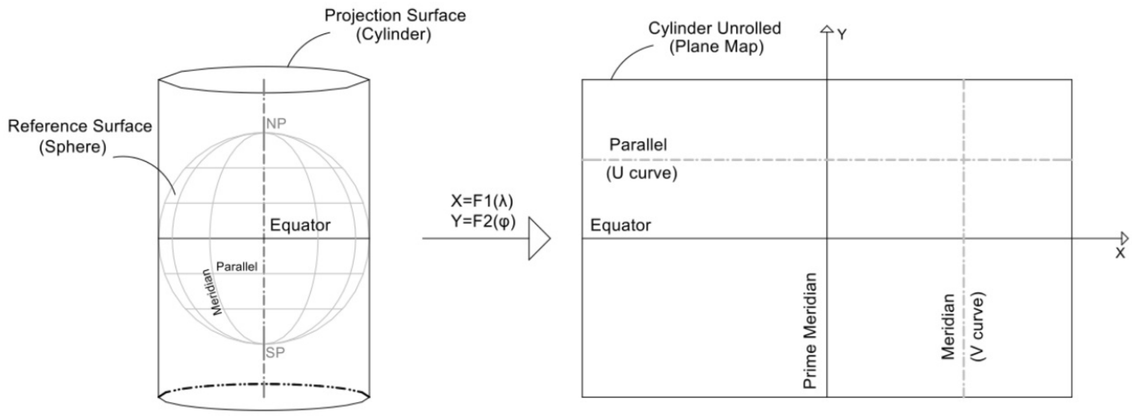

Figure 2 - The general concept of Mercator projection. Source: Authors.

Having made these considerations, from now on the task is to find the functions F1 and F2 that represent the Mercator projection on the sphere, and, additionally, its scale factor.

THE FIRST GAUSSIAN FUNDAMENTAL QUANTITIES

Consider a sphere of radius $R > 0$ in a three-dimensional Cartesian coordinate system, where curved lines in two directions, called parallels and meridians, correspond to constant latitude φ and longitude λ values, respectively, as shown in Figure 3.

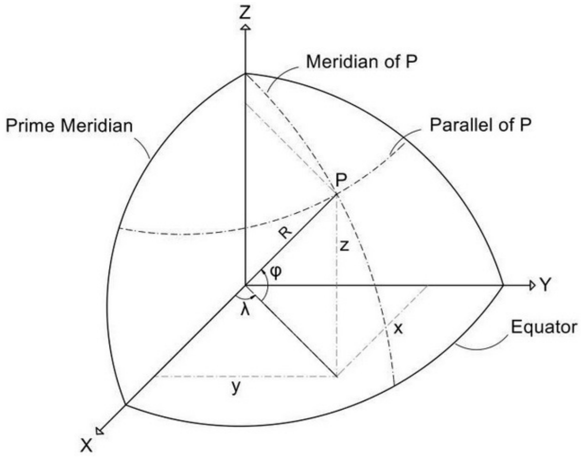

Figure 3 - Parallel of latitude and meridian of longitude of a point P on the sphere. Source: Authors.

According to the geometry of Figure 3, the Cartesian coordinates of any point on this surface can be expressed using the equations (5), (6), and (7):

$$x = R cos\varphi cos\lambda \quad (5)$$

$$y = R cos\varphi sin\lambda \quad (6)$$

$$z = R sin\varphi \quad (7)$$

On the same surface, let P(x,y,z) and Q(x+dx,y+dy,z+dz) be any two points, where dx,dy,dz are infinitesimal elements. Since they are infinitesimal, the linear element ds of a curve between P and Q can be represented by equations (8), (9) and (10) as follows:

$$ds^2 = (x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2 \quad (8)$$

$$ds^2 = (x + dx - x)^2 + (y + dy - y)^2 + (z + dz - z)^2 \quad (9)$$

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (10)$$

The elements (x,y,z) and (dx,dy,dz) have a relationship, which can be found through the total differentials of equations (5), (6) and (7), as shown in equations (11), (12) and (13):

$$dx = \frac{\partial x}{\partial \varphi} d\varphi + \frac{\partial x}{\partial \lambda} d\lambda \quad (11)$$

$$dy = \frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial \lambda} d\lambda \quad (12)$$

$$dz = \frac{\partial z}{\partial \varphi} d\varphi + \frac{\partial z}{\partial \lambda} d\lambda \quad (13)$$

Substituting equations (11), (12) and (13) into equation (10) and expanding, one can obtain equations (14), (15), and (16) as follows:

$$ds^2 = (\frac{\partial x}{\partial\varphi}d\varphi + \frac{\partial x}{\partial\lambda}d\lambda)^2 + (\frac{\partial y}{\partial\varphi}d\varphi + \frac{\partial y}{\partial\lambda}d\lambda)^2 + (\frac{\partial z}{\partial\varphi}d\varphi + \frac{\partial z}{\partial\lambda}d\lambda)^2 \tag{14}$$

$$ds^2 = (\frac{\partial x}{\partial\varphi})^2 d\varphi^2 + \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda}2d\varphi d\lambda + (\frac{\partial x}{\partial\lambda})^2 d\lambda^2 + $$
$$+ (\frac{\partial y}{\partial\varphi})^2 d\varphi^2 + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda}2d\varphi d\lambda + (\frac{\partial y}{\partial\lambda})^2 d\lambda^2 + \tag{15}$$
$$+ (\frac{\partial z}{\partial\varphi})^2 d\varphi^2 + \frac{\partial z}{\partial\varphi}\frac{\partial z}{\partial\lambda}2d\varphi d\lambda + (\frac{\partial z}{\partial\lambda})^2 d\lambda^2$$

$$ds^2 = \left[(\frac{\partial x}{\partial\varphi})^2 + (\frac{\partial y}{\partial\varphi})^2 + (\frac{\partial z}{\partial\varphi})^2\right]d\varphi^2 +$$
$$+\left[\frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda} + \frac{\partial z}{\partial\varphi}\frac{\partial z}{\partial\lambda}\right]2d\varphi d\lambda + \tag{16}$$
$$+\left[(\frac{\partial x}{\partial\lambda})^2 + (\frac{\partial y}{\partial\lambda})^2 + (\frac{\partial z}{\partial\lambda})^2\right]d\lambda^2$$

Equation (16) can be abbreviated as shown in equations (17), (18) and (19), by making:

$$e = (\frac{\partial x}{\partial\varphi})^2 + (\frac{\partial y}{\partial\varphi})^2 + (\frac{\partial z}{\partial\varphi})^2 \tag{17}$$

$$f = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda} + \frac{\partial z}{\partial\varphi}\frac{\partial z}{\partial\lambda} \tag{18}$$

$$g = (\frac{\partial x}{\partial\lambda})^2 + (\frac{\partial y}{\partial\lambda})^2 + (\frac{\partial z}{\partial\lambda})^2 \tag{19}$$

The quantities e,f and g are known as the First Gaussian Fundamental Quantities (RICHARDUS and ADLER, 1972), hereinafter called GQ's. Therefore, an infinitesimal linear element ds of a curve on the surface of a sphere can be represented by equation (20) (DEAKIN, 2003):

$$ds^2 = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \text{17-19} \tag{20}$$

The GQ's equations have partial derivatives that, for the sphere, can be found by deriving equations (5), (6), and (7), according to equations (21) to (26):

$$\frac{\partial x}{\partial\varphi} = \frac{\partial(Rcos\varphi cos\lambda)}{\partial\varphi} =- Rsin\varphi cos\lambda \tag{21}$$

$$\frac{\partial x}{\partial\lambda} = \frac{\partial(Rcos\varphi cos\lambda)}{\partial\lambda} =- Rcos\varphi sin\lambda \tag{22}$$

$$\frac{\partial y}{\partial\varphi} = \frac{\partial(Rcos\varphi sin\lambda)}{\partial\varphi} =- Rsin\varphi sin\lambda \tag{23}$$

$$\frac{\partial y}{\partial\lambda} = \frac{\partial(Rcos\varphi sin\lambda)}{\partial\lambda} = Rcos\varphi cos\lambda \tag{24}$$

$$\frac{\partial z}{\partial\varphi} = \frac{\partial(Rsin\varphi)}{\partial\varphi} = Rcos\varphi \tag{25}$$

$$\frac{\partial z}{\partial\lambda} = \frac{\partial(Rsin\varphi)}{\partial\lambda} = 0 \tag{26}$$

Substituting equations (21) to (26) into equations (17), (18) and (19), yields equations (27) to (39):

$$e = (-Rsin\varphi cos\lambda)^2 + (-Rsin\varphi sin\lambda)^2 + (Rcos\varphi)^2 \tag{27}$$

$$e = R^2sin^2\varphi cos^2\lambda + R^2sin^2\varphi sin^2\lambda + R^2cos^2\varphi \tag{28}$$

$$e = R^2(sin^2\varphi cos^2\lambda + sin^2\varphi sin^2\lambda + cos^2\varphi) \tag{29}$$

$$e = R^2(sin^2\varphi(cos^2\lambda + sin^2\lambda) + cos^2\varphi) = R^2(sin^2\varphi + cos^2\varphi) \tag{30}$$

$$e = R^2 \tag{31}$$

$$f = (-Rsin\varphi cos\lambda)(-Rcos\varphi sin\lambda) + (-Rsin\varphi sin\lambda)(Rcos\varphi cos\lambda) \tag{32}$$

$$f = R^2sin\varphi cos\lambda cos\varphi sin\lambda - R^2sin\varphi sin\lambda cos\varphi cos\lambda \tag{33}$$

$$f = 0 \tag{34}$$

$$g = (-Rcos\varphi sin\lambda)^2 + (Rcos\varphi cos\lambda)^2 \tag{35}$$

$$g = R^2cos^2\varphi sin^2\lambda + R^2cos^2\varphi cos^2\lambda \tag{36}$$

$$g = R^2(cos^2\varphi sin^2\lambda + cos^2\varphi cos^2\lambda) \tag{37}$$

$$g = R^2(cos^2\varphi(sin^2\lambda + cos^2\lambda)) \tag{38}$$

$$g = R^2cos^2\varphi \tag{39}$$

Once one has the methodology for determining the GQ's for the sphere, one can use it to determine the GQ's for the projection surface (normal cylinder), with the only difference being in the nomenclature.

Thus, calling E,F,G the GQ's in relation to equations (3) and (4), and dS an infinitesimal linear element on the surface of the cylinder, one arrive at equations (40) to (43):

$$E = (\frac{\partial X}{\partial \varphi})^2 + (\frac{\partial Y}{\partial \varphi})^2 + (\frac{\partial Z}{\partial \varphi})^2 = 0 + (\frac{\partial Y}{\partial \varphi})^2 + 0 = (\frac{\partial Y}{\partial \varphi})^2 \tag{40}$$

$$F = \frac{\partial X}{\partial \varphi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \varphi}\frac{\partial Y}{\partial \lambda} + \frac{\partial Z}{\partial \varphi}\frac{\partial Z}{\partial \lambda} = 0 + 0 + 0 = 0 \tag{41}$$

$$G = (\frac{\partial X}{\partial \lambda})^2 + (\frac{\partial Y}{\partial \lambda})^2 + (\frac{\partial Z}{\partial \lambda})^2 = (\frac{\partial X}{\partial \lambda})^2 + 0 + 0 = (\frac{\partial X}{\partial \lambda})^2 \tag{42}$$

$$dS^2 = Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2 \tag{43}$$

It should be noted that equations (40) to (43) have elements from both the sphere and the cylinder.

THE INFINITESIMAL QUADRILATERAL ON THE SPHERE

Figure 4 shows an infinitesimal element ds, limited by the curves passing through the point P(φ,λ) and Q(φ+dφ,λ+dλ) on the surface of the sphere.

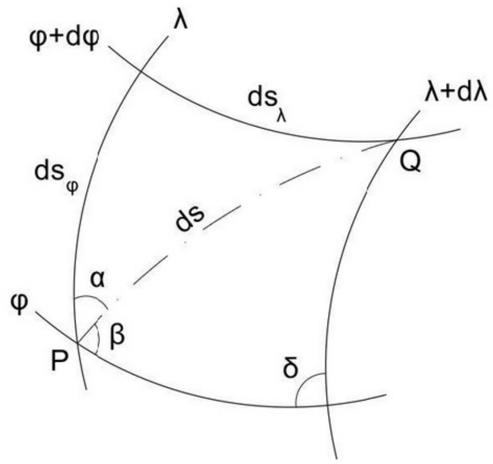

Figure 4 - An infinitesimal quadrilateral on the surface of the sphere. Source: Authors.

Since these are infinitesimal quantities, one can assume that the quadrilateral in Figure 4 is infinitesimal, and that ds is an infinitesimal line segment. Then, the tangents to the curves at P will coincide with the corresponding straight segments (BUGAYEVSKIY and SNYDER, 1995). This means that angle α can be considered as the azimuth from P to Q. Regarding the elements ds_φ and ds_λ they can be initially represented by the general equations (44) and (45):

$$ds_\varphi^2 = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \quad (44)$$

$$ds_\lambda^2 = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \quad (45)$$

Since the value of is constant along a parallel, this also occurs along ds. Therefore, along this section, dφ=0. The same reasoning applies to the meridian represented by , resulting in dλ=0 along ds. Therefore, equations (44) and (45) become equations (46) to (49):

$$ds_\varphi^2 = ed\varphi^2 \quad (46)$$

$$ds_\lambda^2 = gd\lambda^2 \quad (47)$$

$$ds_\varphi = \sqrt{e}d\varphi \quad (48)$$

$$ds_\lambda = \sqrt{g}d\lambda \quad (49)$$

Since they are differential elements, point Q is infinitesimally close to point P. Therefore, the rules of plane trigonometry can be used. Because of this, the angle can consider the plane azimuth from

P to Q (SANTOS, 1985). Thus, at point P, the slope of ds can be written according to equation (50):

$$tan(\alpha) = \lim_{Q\to P} \frac{ds_\lambda}{ds_\varphi} = \lim_{Q\to P} \frac{\sqrt{g}d\lambda}{\sqrt{e}d\varphi} = \frac{\sqrt{g}d\lambda}{\sqrt{e}d\varphi} \quad (50)$$

From the differential elements of Figure 4, the sum of the quadrilateral´s internal angles is equal to 2π (or 360°). By making $\omega = \alpha + \beta$, one can write equations (51), (52) and (53):

$$2\omega + 2\delta = 2\pi \quad (51)$$

$$\omega + \delta = \pi \quad (52)$$

$$\delta = \pi - \omega \quad (53)$$

Now, by the Law of Cosines, one can write equations (54), (55) and (56):

$$ds^2 = (\sqrt{g}d\lambda)^2 + (\sqrt{e}d\varphi)^2 - 2\sqrt{g}d\lambda\sqrt{e}d\varphi(cos(\delta)) \quad (54)$$

$$ds^2 = gd\lambda^2 + ed\varphi^2 - 2\sqrt{g}d\lambda\sqrt{e}d\varphi(cos(\pi - \omega)) \quad (55)$$

$$ds^2 = gd\lambda^2 + ed\varphi^2 + 2\sqrt{g}d\lambda\sqrt{e}d\varphi(cos(\omega)) \quad (56)$$

Equating equations (20) and (56) and developing them, one obtains equations (57) to (61):

$$ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 = gd\lambda^2 + ed\varphi^2 + 2\sqrt{g}d\lambda\sqrt{e}d\varphi(cos(\omega)) \quad (57)$$

$$2fd\varphi d\lambda = gd\lambda^2 - gd\lambda^2 + ed\varphi^2 - ed\varphi^2 + 2\sqrt{g}d\lambda\sqrt{e}d\varphi(cos(\omega)) \quad (58)$$

$$2fd\varphi d\lambda = 2\sqrt{g}d\lambda\sqrt{e}d\varphi(cos(\omega)) \quad (59)$$

$$f = \sqrt{g}\sqrt{e}(cos(\omega)) \quad (60)$$

$$cos(\omega) = \frac{f}{\sqrt{ge}} \quad (61)$$

Equation (61) defines the angle formed between a parallel and a meridian. Since, for the sphere, in equation (34) it has already been deduced that f=0, then the cosine of the angle between these two lines is equal to zero. It is, therefore, an orthogonal intersection between the parallels and the meridians.

THE SCALE FACTOR

According to (KRAKIWSKY, 1973, p. 30), “the scale factor describes, at each point on the map projection, the amount of distortion in length. This distortion is of course due to maintaining conformality and fulfilling other conditions prescribed for the projection”.

To find the scale factor, denoted here as m, for any point P on the projection surface, it is necessary to determine the relationship between the linear differential elements on the projection surface and those on the reference surface. Thus, when a point Q is infinitesimally close to point P, one can write Equation (62) (OSBORNE, 2013):

$$m = \lim_{Q \to P} \frac{dS}{ds} = \frac{dS}{ds} \tag{62}$$

Since the equations for dS and ds have already been derived respectively in equations (43) and (20), and considering that f=F=0, one can write equation (63):

$$m^2 = \frac{dS^2}{ds^2} = \frac{Ed\varphi^2+2Fd\varphi d\lambda+Gd\lambda^2}{ed\varphi^2+2fd\varphi d\lambda+gd\lambda^2} = \frac{Ed\varphi^2+Gd\lambda^2}{ed\varphi^2+gd\lambda^2} \tag{63}$$

To express equation (63) in terms of the rate of change, one can divide both the numerator and denominator on the right-hand side of the equation by d2

, obtaining equation (64):

$$m^2 = \frac{E(\frac{d\varphi}{d\lambda})^2+G}{e(\frac{d\varphi}{d\lambda})^2+g} \tag{64}$$

The rate of change in equation (64) can be found as follows. It was seen from equation (50) that, for the sphere, one can write equation (65):

$$tan(\alpha) = \frac{\sqrt{g}d\lambda}{\sqrt{e}d\varphi} \tag{65}$$

Developing equation (65), one can obtain the equations (66) and (67):

$$\frac{d\varphi}{d\lambda} = \frac{\sqrt{g}}{tan(\alpha)\sqrt{e}} \tag{66}$$

$$(\frac{d\varphi}{d\lambda})^2 = \frac{g}{e(tan^2\alpha)} \tag{67}$$

Substituting equation (67) into equation (64) and developing, one arrives at equation (68):

$$m^2 = \frac{E\frac{g}{e(tan^2\alpha)}+G}{e\frac{g}{e(tan^2\alpha)}+g} = \frac{E\frac{g}{e(tan^2\alpha)}+G}{\frac{g}{tan^2\alpha}+g} = \frac{E\frac{g}{e(tan^2\alpha)}+G}{g(\frac{1}{tan^2\alpha}+1)} = \frac{E\frac{g}{e(tan^2\alpha)}+G}{g(\frac{cos^2\alpha}{sin^2\alpha}+1)} = \frac{E\frac{g}{e(tan^2\alpha)}+G}{g(\frac{1}{sin^2\alpha}(cos^2\alpha+sin^2\alpha))} =$$

$$= \frac{E\frac{g}{e(tan^2\alpha)}+G}{\frac{g}{sin^2\alpha}} = (E\frac{g}{e(tan^2\alpha)} + G)\frac{sin^2\alpha}{g} = \frac{E}{e}\frac{sin^2\alpha}{tan^2\alpha} + \frac{G}{g}sin^2\alpha = \tag{68}$$

$$= \frac{E}{e}cos^2\alpha + \frac{G}{g}sin^2\alpha$$

On a sphere, parallel lines are perpendicular to meridian lines, so any two points lying on the same meridian will have an azimuth of either 0 or π, resulting in a sine value of zero. Similarly, any two points lying on the same parallel will have an azimuth of either π/2 or 3π/2, resulting in a cosine value of zero.

These considerations give rise to two special scale factors: the meridional scale factor m_m and the parallel scale factor m_p, which can be derived from Equation (68) and are given in equations (69) to (72):

$$m_m^2 = \frac{E}{e}(1) + \frac{G}{g}(0) = \frac{E}{e} \tag{69}$$

$$m_m = \frac{\sqrt{E}}{\sqrt{e}} \tag{70}$$

$$m_p^2 = \frac{E}{e}(0) + \frac{G}{g}(1) = \frac{G}{g} \tag{71}$$

$$m_p = \frac{\sqrt{G}}{\sqrt{g}} \tag{72}$$

Here lies the origin of a condition that ensures compliance: if mm is equal to mp

, equation (68) becomes independent of angle alpha. To demonstrate this, first consider the equality between
them

as given in equations (73), and (74):

$$m_m = m_p = \frac{\sqrt{E}}{\sqrt{e}} = \frac{\sqrt{G}}{\sqrt{g}} \tag{73}$$

$$m_m^2 = m_p^2 = \frac{E}{e} = \frac{G}{g} \tag{74}$$

In equation (74), as they are equal, their elements will be called K, according to equation (75):

$$m_m^2 = m_p^2 = \frac{E}{e} = \frac{G}{g} = K \tag{75}$$

By substituting equation (75) into equation (68), one obtains equation (76):

$$m^2 = K\cos^2\alpha + K\sin^2\alpha = K(\cos^2\alpha + \sin^2\alpha) = K \tag{76}$$

Thus, equation (76) concludes the demonstration that, under the condition given by equation (75), the scale factor for a point P is independent of direction. Under this condition, it has the same value in the projection for any direction relative to a given point. This type of projection is called a conformal projection. According to (KRAKIWSKY, 1973, p. 01), “conformal map projections are the class of projections in which angles on the surface to be mapped are preserved, that is, corresponding angles on the map plane and the surface are equal”.

For cylindrical projections on the sphere, from equations (75) and (76), this condition can also be

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expressed by equation (77):

$$m^2 = m_m^2 = m_p^2 = \frac{E}{e} = \frac{G}{g} = K \quad (77)$$

Conformal projections maintain a consistent scale factor in all directions at any given point, preserving the shape of objects. This shape preservation also ensures that angles remain unchanged, meaning the angle between two lines originating from a point on the reference surface is identical to the angle between their corresponding projections. However, this property has a restriction: it strictly applies to infinitesimal areas, as determined by the relationship between the infinitesimal elements ds and dS in equation (62).

Therefore, the conformal property holds only for small areas. It is impossible to construct a conformal projection that accurately represents extensive regions without distortion, as a sphere cannot be mapped onto a plane without altering its shape. Consequently, for larger areas, the scale factor changes depending on the position of the points.

In summary, conformity refers to the uniformity of the scale factor in all directions at a specific point. However, its value varies depending on the point's location. This variation in the scale factor from one point to another will be demonstrated below for the Mercator projection.

THE EQUATIONS OF THE MERCATOR PROJECTION

To obtain the equations for the X and Y coordinates, and for the scale factor in the Mercator projection, initially the GQ’s e, g, E and G of equations (31), (39), (40) and (42) will be substituted into equations (70) and (72), to obtain equations (78) and (79):

$$m_m = \frac{\sqrt{E}}{\sqrt{e}} = \frac{\sqrt{\left(\frac{\partial Y}{\partial \varphi}\right)^2}}{\sqrt{R^2}} = \frac{1}{R}\frac{\partial Y}{\partial \varphi} \quad (78)$$

$$m_p = \frac{\sqrt{G}}{\sqrt{g}} = \frac{\sqrt{\left(\frac{\partial X}{\partial \lambda}\right)^2}}{\sqrt{R^2 cos^2\varphi}} = \frac{1}{Rcos\varphi}\frac{\partial X}{\partial \lambda} \quad (79)$$

For cylindrical projections, the line of tangency is a single true length line (PEARSON, 1990). In the Mercator projection, the scale factor at the equator is equal to 1, due to the tangency of the sphere with the cylinder. In this case, this scale factor is usually called $k_0 = 1$. Therefore, knowing that the latitude at the equator is $\varphi_0 = 0°$, with equation (79) one can write equations (80) and (81):

$$k_0 = \frac{1}{R}\frac{\partial X}{\partial \lambda} = 1 \quad (80)$$

$$R = \frac{\partial X}{\partial \lambda} \quad (81)$$

To solve equation (81) one can proceed as follows. As seen in equation (3), in cylindrical projections, X is only a function of λ. This obviously means that X is not a function of φ. This information is very important, since it means that the total differential of X can be written as in equation (82):

$$dX = \frac{\partial X}{\partial \lambda} d\lambda \tag{82}$$

Therefore, from equation (82), in this case, one can write equation (83):

$$\frac{dX}{d\lambda} = \frac{\partial X}{\partial \lambda} \tag{83}$$

Substituting equation (83) into equation (81), one can write equation (84):

$$R = \frac{\partial X}{\partial \lambda} = \frac{dX}{d\lambda} \tag{84}$$

Isolating dX in equation (84) one arrives at equation (85):

$$dX = Rd\lambda \tag{85}$$

To find X, one can integrate, according to equations (86) and (87):

$$\int dx = \int Rd\lambda \tag{86}$$

$$X = R\lambda + C^{t_1} \tag{87}$$

The constant C^{t_1} in equation (87) can be calculated as follows. In Mercator projection, when $X = 0$, there will be a corresponding value for λ, which will be called λ_0. Thus, the constant can be isolated, as per equations (88) and (89):

$$R\lambda_0 + C^{t_1} = 0 \tag{88}$$

$$C^{t_1} = - R\lambda_0 \tag{89}$$

Substituting equation (89) into equation (87), one obtains equations (90) and (91) (SNYDER, 1987; BUGAYEVSKIY and SNYDER, 1995):

$$X = R\lambda - R\lambda_0 \tag{90}$$

$$X = R(\lambda - \lambda_0) \tag{91}$$

Equation (91) represents the X coordinate in the Mercator projection. In this equation, R is the radius of the sphere at the scale of the map as drawn, λ is the longitude of the point in radians, and the value of λ_0 can be arbitrary, in radians, so that the meridian of origin passes through the area of interest. For example, if $\lambda_0 = 0$, the meridian of origin will coincide with the Greenwich meridian.

The inverse relationship for equation (91) can be obtained by isolating λ, which results in equation (92):

$$\lambda = \frac{X}{R} + \lambda_0 \tag{92}$$

To find an equation for Y, the procedure will be as follows. The starting point will be equation (73), which allows the equality between equations (78) and (79). By making this equality, one arrive at equation (93):

$$m_m = m_p = \frac{1}{R}\frac{\partial Y}{\partial \varphi} = \frac{1}{R\cos\varphi}\frac{\partial X}{\partial \lambda} \Rightarrow \frac{\partial Y}{\partial \varphi} = \frac{1}{\cos\varphi}\frac{\partial X}{\partial \lambda} \tag{93}$$

As seen in equation (4), Y is dependent only on . Thus, in a manner analogous to what was done for X, it can also be done for Y, resulting in equation (94):

$$\frac{dY}{d\varphi} = \frac{\partial Y}{\partial \varphi} \tag{94}$$

Substituting equations (94), (83) and (84) into equation (93), and solving for dY, one arrives at equations (95) and (96):

$$\frac{dY}{d\varphi} = \frac{1}{\cos\varphi}\frac{dx}{d\lambda} = \frac{1}{\cos\varphi}R \tag{95}$$

$$dY = \frac{R}{\cos\varphi}d\varphi \tag{96}$$

Integrating equation (96) and solving, one obtains equations (97) to (104):

$$\int dY = \int \frac{R}{cos\varphi} d\varphi \tag{97}$$

$$Y = R\int \frac{1}{cos\varphi} d\varphi = R\int sec\varphi d\varphi \tag{98}$$

$$Y = R\int sec\varphi \frac{sec\varphi + tan\varphi}{sec\varphi + tan\varphi} d\varphi \tag{99}$$

$$\zeta = sec\varphi + tan\varphi \tag{100}$$

$$d\zeta = (sec\varphi tan\varphi + sec^2\varphi)d\varphi \tag{101}$$

$$Y = R\int \frac{1}{\zeta} d\zeta \tag{102}$$

$$Y = Rln\zeta + C^{t_2} \tag{103}$$

$$Y = Rln(sec\varphi + tan\varphi) + C^{t_2} \tag{104}$$

The constant C^{t_2} in equation (104) can be calculated as follows. In the Mercator projection, when $Y = 0$, it occurs that $\varphi = 0°$. Thus, from equation (104), the constant can be isolated, according to equations (105), (106) and (107):

$$0 = Rln(sec(0) + tan(0)) + C^{t_2} \tag{105}$$

$$0 = Rln(1) + C^{t_2} \tag{106}$$

$$C^{t_2} = 0 \tag{107}$$

Now, substituting equation (107) into equation (104), one obtains equation (108):

$$Y = Rln(sec\varphi + tan\varphi) \tag{108}$$

Equation (108) can be solved by means of known trigonometric identities. The first two will be the identities represented by equations (109) and (110):

$$sec\varphi = \frac{1}{cos\varphi} \tag{109}$$

$$tan\varphi = \frac{sin\varphi}{cos\varphi} \tag{110}$$

Substituting equations (109) and (110) into equation (108) and expanding, one obtains equation (111):

$$Y = Rln(\frac{1}{cos\varphi} + \frac{sin\varphi}{cos\varphi}) = R\ ln(\frac{1+sin\varphi}{cos\varphi}) \quad (111)$$

Equation (111) presents two known trigonometric identities, according to equations (112) and (113):

$$1 + sin\varphi = 2cos^2(\frac{\pi}{4} + \frac{\varphi}{2}) \quad (112)$$

$$cos\varphi = 2cos(\frac{\pi}{4} + \frac{\varphi}{2})cos(\frac{\pi}{4} - \frac{\varphi}{2}) \quad (113)$$

Substituting equations (112) and (113) into equation (111), and expanding, one arrives at equation (114):

$$Y = Rln[\frac{2cos^2(\frac{\pi}{4}+\frac{\varphi}{2})}{2cos(\frac{\pi}{4}+\frac{\varphi}{2})cos(\frac{\pi}{4}-\frac{\varphi}{2})}] = Rln[\frac{cos(\frac{\pi}{4}+\frac{\varphi}{2})}{cos(\frac{\pi}{4}-\frac{\varphi}{2})}] \quad (114)$$

Equation (114) presents another known trigonometric identity, according to equation (115):

$$\frac{cos(\frac{\pi}{4}+\frac{\varphi}{2})}{cos(\frac{\pi}{4}-\frac{\varphi}{2})} = tan(\frac{\pi}{4} + \frac{\varphi}{2}) \quad (115)$$

Substituting equation (115) into equation (114) one finds equation (116) (SNYDER, 1987; BUGAYEVSKIY and SNYDER, 1995):

$$Y = Rln[tan(\frac{\pi}{4} + \frac{\varphi}{2})] \quad (116)$$

Equation (116) represents the Y coordinate in the Mercator projection, in which R is the radius of the sphere at the scale of the map as drawn, and φ is the latitude, in radians, of the point.

In cartographic literature, the term $ln[tan(\frac{\pi}{4} + \frac{\varphi}{2})]$ is called isometric latitude, representing by q (LAPAINE, 2024; MALING, 1992), or by ψ (VERMEER and RASILA, 2020; SNYDER, 1987), or even by χ (REDFEARN, 1948; LEE, 1945).

The inverse relationship of the equation (116) can be found by isolating φ, as shown in equations (117) to (120):

$$\frac{Y}{R} = ln[tan(\frac{\pi}{4} + \frac{\varphi}{2})] \quad (117)$$

$$e^{\frac{Y}{R}} = tan(\frac{\pi}{4} + \frac{\varphi}{2}] \quad (118)$$

$$atan(e^{\frac{Y}{R}}) = \frac{\pi}{4} + \frac{\varphi}{2} \quad (119)$$

$$\varphi = 2[atan(e^{\frac{Y}{R}}) - \frac{\pi}{4}] \quad (120)$$

Equations (91) and (116) can be used to find the GQ's for the Mercator projection, substituting them in equations (40), (41) and (42), to obtain equations (121), (122) and (123):

$$E = (\frac{\partial Y}{\partial \varphi})^2 = \{\frac{\partial}{\partial \varphi} Rln[tan(\frac{\pi}{4} + \frac{\varphi}{2})]\}^2 \quad (121)$$

$$F = 0 \quad (122)$$

$$G = (\frac{\partial X}{\partial \lambda})^2 = [\frac{\partial}{\partial \lambda} R(\lambda - \lambda_0)]^2 = R^2 \quad (123)$$

To find E, it will be necessary to calculate the partial derivative in equation (121). It can be calculated as follows. First, note that Y is a composite function. Thus, to facilitate the calculations, a first substitution will be made in equation (121), according to equations (124) and (125):

$$i = \frac{\pi}{4} + \frac{\varphi}{2} \quad (124)$$

$$Y = Rln[tan(i)] \quad (125)$$

With equation (124) one can obtain the first partial derivative relevant to the solution, represented by equation (126):

$$\frac{\partial i}{\partial \varphi} = \frac{1}{2} \quad (126)$$

The next step is to make the second substitution, this time in equation (125), according to equations (127) and (128):

$$j = tan(i) \quad (127)$$

$$Y = Rln(j) \quad (128)$$

With equations (127) and (128) one can obtain the last two partial derivatives relevant to the solution, represented by equations (129) and (130):

$$\frac{\partial j}{\partial i} = sec^2 i \quad (129)$$

$$\frac{\partial Y}{\partial j} = \frac{R}{j} \quad (130)$$

Now, since Y is a composite function, using equations (126), (129) and (130), by the Chain Rule one can write equation (131):

$$\frac{\partial Y}{\partial \varphi} = \frac{\partial Y}{\partial j}\frac{\partial j}{\partial i}\frac{\partial i}{\partial \varphi} = \frac{R}{j}sec^2(i)\frac{1}{2} = \frac{R}{tan(i)}sec^2(i)\frac{1}{2} = R\frac{cos(i)}{sin(i)}\frac{1}{cos^2(i)}\frac{1}{2} =$$

$$= R\frac{1}{2sin(i)cos(i)} = R\frac{1}{sin(2i)} = R\frac{1}{sin[2(\frac{\pi}{4}+\frac{\varphi}{2})]} = R\frac{1}{sin(\frac{\pi}{2}+\varphi)} = \quad (131)$$

$$= R\frac{1}{cos\varphi}$$

Substituting equation (131) into equation (121), one arrives at equation (132):

$$E = (\frac{\partial Y}{\partial \varphi})^2 = \frac{R^2}{cos^2\varphi} \quad (132)$$

Regarding the scale factor for the Mercator projection, one can proceed as follows. Initially one can use equation (77), apply the square root to all terms, and then leave it as in equation (133):

$$m = m_m = m_p = \frac{\sqrt{E}}{\sqrt{e}} = \frac{\sqrt{G}}{\sqrt{g}} \quad (133)$$

The GQ's from equation (133) can be found in equations (31), (39), (123) and (132). Substituting them separately into equation (133) yields equations (134) and (135), confirming that $m = m_m = m_p$.

Equations (134) and/or (135) shows that the scale factor for the Mercator projection varies with the inverse cosine of the latitude. This illustrates, for example, the increasing distortion as one approaches the poles, which makes this projection unsuitable for mapping these regions. Furthermore, according to Osborne (2013, p. 32), “conformality implies isotropy of scale: meridian scale, parallel scale and general scale are all equal to secφ in the Mercator projection”.

VISUALIZATION OF THE MERCATOR PROJECTION

Equations (91) and (116) are the plotting equations for Mercator projection on the sphere (PEARSON, 1990). As an example, a Python program was written to plot the visualization of the continent contours between latitudes of -80° and +80° starting on the equator, with Greenwich as prime meridian. Figure 5 shows the result, and Annex 1 contains the script.

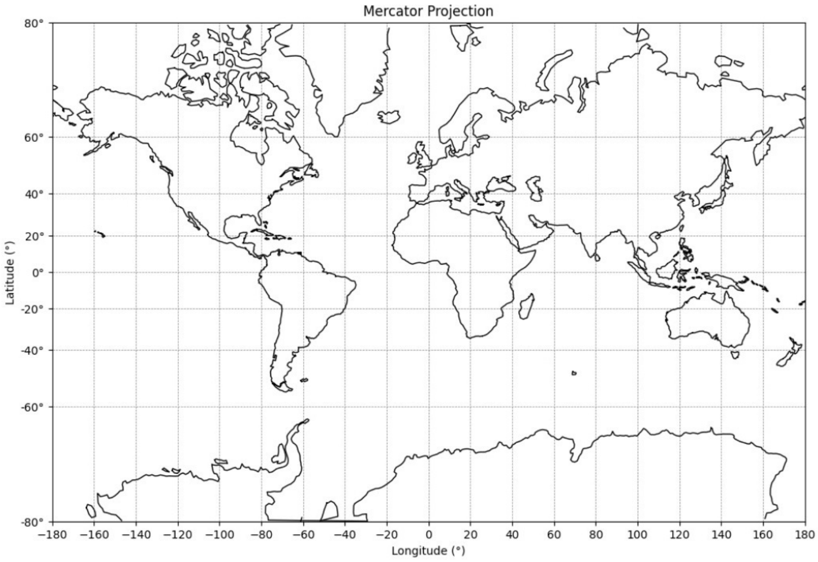

Figure 5 - A Mercator projection visualization. Source: Authors.

After viewing, and still with the equations in mind, one can see that:

> "The Mercator projection is a normal cylindrical projection, with the cylinder conceptually tangent to the equator. Lines of constant scale follow the parallels of latitude, all of which are straight and run parallel to the equator, except for the poles, which are at infinity. The scale on the Mercator increases away from the equator. The projection is conformal, and it is recommended for large-scale conformal mapping of regions bordering the equator" (SNYDER, 1997, p. 424).

It should also be noted that the intervals between the parallels increase proportionally to secφ (RICHARDUS and ADLER, 1974). Furthermore, the equation (91) shows that the longitude is transformed linearly into X without any deformation. This means that the meridians, which originally converge at the poles on the sphere, are projected as equally spaced vertical lines on the projection plane.

There is another feature of the Mercator projection, that is a consequence of its equations, that due to its importance, should be highlighted. Thus,

> "Picture yourself as a seventeenth-century navigator who knows where he is and where he wants to go. You plot both locations on a chart, join them with a straight line, and measure the angle your line makes with the map's meridians, which run due north. If the chart is a Mercator map, all its meridians are straight lines, parallel to one another, and the course you've just plotted is a rhumb line, also called a loxodrome" (MONMONIER, 2004, p. 01).

According to Bugayevskiy and Snyder (1995, p. 64), "only the Mercator projection shows all loxodromes as straight lines". However, they are generally not the shortest path between two points. Despite this, loxodromes were widely used for planning flight routes until the 1960's, when they were replaced by great-circle routes for efficiency (VERMEER and RASILA, 2020).

CONCLUSION

The mathematical foundations of map projections are inherently complex. However, a solid understanding of this subject is essential for making more consistent decisions in various problems related to cartography, geodesy, and geospatial mapping. A common challenge in the literature, in general, is the presence of mathematical derivations that often include implicit steps, making comprehension difficult for readers lacking an advanced mathematical background. This can lead to increased difficulty and, in some cases, discouragement.

This article aims to provide a rigorous and comprehensive mathematical equations derivation of the Mercator projection, which is one of the most historically significant and widely used projections in cartography. The methodology integrates both mathematical cartography and a structured pedagogical approach to facilitate understanding. Particular emphasis has been placed on ensuring continuity in derivations, minimizing omissions, and maintaining a logical progression of equations to mitigate potential conceptual gaps.

Additionally, the article presents an original Python implementation for visualizing the Mercator projection. This serves two primary purposes: first, to illustrate the projection's fundamental purpose, which is map generation, and second, to demonstrate that proficiency in mathematical cartography, combined with programming skills, enables the development of custom computational tools. This latter aspect is particularly relevant for validating contemporary mapping software and ensuring the accuracy of computational implementations in geospatial applications.

APPENDICES

```python
import numpy as np
import matplotlib.pyplot as plt
import cartopy.io.shapereader as shpreader
from shapely.geometry import LineString, MultiLineString
def mercator(lon, lat, R=180/np.pi):
    lon = np.asarray(lon)
    lat = np.asarray(lat)
    lon_rad = np.deg2rad(lon)
    lat_rad = np.deg2rad(lat)
    x = R * lon_rad
    y = R * np.log(np.tan(np.pi/4 + lat_rad/2))
    x, y = np.broadcast_arrays(x, y)
    return x, y
plt.figure(figsize=(12, 8))
lat_min, lat_max = -80, 80
lon_min, lon_max = -180, 180
meridianos = np.arange(lon_min, lon_max+1, 20)
lat_vals = np.linspace(lat_min, lat_max, 300)
for lon in meridianos:
    x, y = mercator(lon, lat_vals)
    plt.plot(x, y, color='gray', linestyle='--', linewidth=0.5)
paralelos = np.arange(lat_min, lat_max+1, 20)
lon_vals = np.linspace(lon_min, lon_max, 300)
for lat in paralelos:
    x, y = mercator(lon_vals, lat)
    plt.plot(x, y, color='gray', linestyle='--', linewidth=0.5)
shpfilename = shpreader.natural_earth(resolution='110m',
                                      category='physical',
                                      name='coastline')
reader = shpreader.Reader(shpfilename)
for record in reader.records():
    geom = record.geometry
    def plot_geom(coords):
        coords = np.array(coords)
        mask = (coords[:, 1] >= lat_min) & (coords[:, 1] <= lat_max)
        if np.sum(mask) < 2:
            return
        lon_line = coords[mask, 0]
        lat_line = coords[mask, 1]
        x, y = mercator(lon_line, lat_line)
        plt.plot(x, y, color='black', linewidth=1)
    if geom.geom_type == 'MultiLineString':
        for line in geom:
            plot_geom(line.coords)
    elif geom.geom_type == 'LineString':
        plot_geom(geom.coords)
    else:
        pass
x_min, _ = mercator(lon_min, 0)
x_max, _ = mercator(lon_max, 0)
plt.xlim(x_min, x_max)
_, y_min = mercator(0, lat_min)
_, y_max = mercator(0, lat_max)
plt.ylim(y_min, y_max)
plt.xticks(np.arange(lon_min, lon_max+1, 20))
lat_ticks = np.arange(lat_min, lat_max+1, 20)
y_ticks = [mercator(0, lat)[1] for lat in lat_ticks]
plt.yticks(y_ticks, [f"{lat}°" for lat in lat_ticks])
plt.xlabel("Longitude (°)")
plt.ylabel("Latitude (°)")
plt.title("Mercator Projection")
plt.grid(False)
plt.show()
```

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